Ch 3. Ex 15

Show that f is α-Hölder $\Rightarrow$ $\hat{f}(n) = O\left(\frac{1}{|n|^\alpha}\right)$

(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}\,\mathrm{d}x$$

hence

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}\,\mathrm{d}x.$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})e^{-in(y+\frac{\pi}{n})}\,dy \qquad \left(x = y+\pi/n\right)$$

$$= \frac{-1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})\,e^{-iny}\,dy$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+\tfrac{\pi}{n})\,e^{-iny}\,dy$$

(since $f(y+\frac{\pi}{n})e^{-iny}$ is 2π-periodic)

$$\hat{f}(n) = \frac{1}{2}\left(\hat{f}(n) + \hat{f}(n)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n}) e^{-inx} dx\right)$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x) - f(x+\tfrac{\pi}{n})\right) e^{-inx} dx$$

(b) Now assume that f satisfies a Hölder condition of order α, namely

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for some $0 < \alpha \leq 1$, some $C > 0$, and all x, h. Use part (a) to show that

$$\hat{f}(n) = O(1/|n|^\alpha).$$

From (a),

$$|\hat{f}(n)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} | f(x) - f(x+\tfrac{\pi}{n}) | dx$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} C \cdot \left|\frac{\pi}{n}\right|^{2\alpha} dx$$

$$= \frac{\pi^{2\alpha}}{2} C \frac{1}{|n|^{2\alpha}}$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies

$$|f(x+h) - f(x)| \le C|h|^\alpha,$$

and $\hat{f}(N) = 1/N^\alpha$ whenever $N = 2^k$.

This shows that $\exists$ α-Hölder f s.t.

$\hat{f}(n) \neq \underbrace{o\left(\frac{1}{|n|^\alpha}\right)}$ small o means $\lim_{n\to\infty} |n|^\alpha \hat{f}(n) = 0$

We have shown that $\hat{f}(n) = \underbrace{O\left(\frac{1}{|n|^\alpha}\right)}$

Big O means $\left(|n|^\alpha \hat{f}(n)\right)_{n=1}^{\infty}$ is a bounded sequence

$$|f(x+h) - f(x)| = \left| \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x} \left(e^{i2^k h} - 1 \right) \right|$$

$$\le \sum_{k=0}^{\infty} 2^{-k\alpha} \left| e^{i2^k h} - 1 \right|$$

$$= \sum_{k=0}^{\infty} 2^{-k\alpha} \left| e^{i2^{k-1} h} \left(e^{i2^{k-1} h} - e^{-i2^{k-1} h} \right) \right|$$

$$= \sum_{k=0}^{\infty} \underbrace{2^{-k\alpha}} \cdot \underbrace{2 \left| \sin(2^{k-1} h) \right|}$$

: A good bound when k is large

: Useful only when $2^{k-1}h$ is small

$$\left(|\sin x| \le |x| \right)$$ Otherwise, we simply bound $|\sin x|$ by 1

We split the sum into two parts.

with indexes k

$$A = \left\{ k : 2^k \le 1/|h| \right\}, \quad B = \left\{ k : 2^k > 1/|h| \right\}$$

$$\sum_{k \in A} 2^{-k\alpha} \, 2 \, |\sin(2^{k-1} h)| \le 2 \cdot \sum_{k \in A} 2^{-k\alpha} \cdot 2^{k-1} |h|$$

$$= |h| \sum_{k \in A} 2^{k(1-\alpha)}$$

Say $2^{N-1} \le 1/|h| < 2^N$

i.e. $A = \{1, 2, \ldots, N-1\}$

$$= |h| \frac{2^{N(1-\alpha)} - 1}{2^{1-\alpha} - 1} \le \frac{2^{N(1-\alpha)} |h|}{2^{1-\alpha} - 1}$$

$$= \frac{2^{1-\alpha}}{2^{1-\alpha}-1} \cdot (2^{N-1})^{1-\alpha} |h|$$

$$= \frac{2^{1-\alpha}}{2^{1-\alpha}-1} \cdot \frac{|h|}{|h|^{1-\alpha}} = C_\alpha |h|^\alpha \left(\because 2^{N-1} \leq 1/|h| \right)$$

$$\sum_{k\in B} 2^{-k\alpha} 2 \cdot |\sin(2^{k-1}h)| \leq 2 \sum_{k\in B} 2^{-k\alpha}$$

Recall that $B = \{N, N+1, \dots\}$

$$= 2 \cdot \frac{2^{-N\alpha}}{1-2^{-\alpha}}$$

$$< \frac{2}{1-2^{-\alpha}} \cdot |h|^\alpha$$

In conclusion,

$$|f(x+h) - f(x)| \leq C_\alpha |h|^\alpha$$

i.e. f is α-Hölder

Moreover, clearly

$$\hat{f}(2^k) = 2^{-k\alpha} = \frac{1}{(2^k)^\alpha} \quad \text{for } k \geq 1$$

Aim to show that if

f is α- Hölder cont for $\alpha > \frac{1}{2}$, then

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

In particular, $S_N f \rightrightarrows f$

(Ex 16 in Ch.3 , Assignment 5)

2. Let f be a α-Hölder continuous function on the circle $[0, 2\pi]$.

   (a) For every positive h, we defined $g_h(x) = f(x+h) - f(x-h)$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2 \, |\hat{f}(n)|^2,$$

   and show that

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 \, |\hat{f}(n)|^2 \leq K^2 h^{2\alpha}.$$

Let $f_h(x) = f(x+h)$.

Then, $\hat{f_h}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x+h) e^{-inx} \, dx$

$\because f(y)e^{-iny}$ is 2π- periodic

$$= \frac{1}{2\pi}\int_h^{2\pi+h} f(y) e^{-iny} \cdot e^{inh} \, dx$$

$$= e^{inh} \hat{f}(n)$$

$\therefore \quad \hat{f}_{-h}(n) = e^{-inh} \hat{f}(n)$

$\therefore \quad$ For $\quad g_h(x) = f_h - f_{-h}$

$$\hat{g}_h(n) = (e^{inh} - e^{-inh}) \hat{f}(n)$$

By Parseval's identity,

$$\frac{1}{2\pi} \int_0^{2\pi} |g_h(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |\hat{g}_h(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |e^{inh} - e^{-inh}| \, |\hat{f}(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} 4 \, |\sin nh|^2 \, |\hat{f}(n)|^2$$

Estimating $\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx$:

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx \leq \frac{1}{2\pi}\int_0^{2\pi} K^2 (2h)^{2\alpha}\,dx$$

$$= 4^{\alpha} \cdot K^2 h^{2\alpha}$$

$$\leq 4 K^2 h^{2\alpha}$$

$$\therefore \sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq K^2 \cdot h^{2\alpha}$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{2K^2\pi^{2\alpha}}{(2^{2p+2})^{\alpha}}$$

(c) Estimate $\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly.

For $h = \frac{\pi}{2^{p+1}}$ and $2^{p-1} < |n| \leq 2^p$,

$$\frac{\pi}{4} < |nh| \leq \frac{\pi}{2}$$

$$\therefore \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right) \leq |\sin nh|$$

$$\sum_{2^{p-1} < |n| \le 2^p} \frac{1}{2} |\hat{f}(n)|^2 \le \sum_{2^{p-1} < |n| \le 2^p} |\sin nh|^2 |\hat{f}(n)|^2$$

$$\le K^2 \cdot h^{2\alpha}$$

$$= K^2 \cdot \frac{\pi^{2\alpha}}{2^{2(p+1)\alpha}}$$

i.e. $$\sum_{2^{p-1} < |n| \le 2^p} |\hat{f}(n)|^2 \le \frac{2K^2 \pi^{2\alpha}}{2^{2(p+1)\alpha}}$$

By Cauchy-Schwarz inequality,

$$\sum_{2^{p-1} < |n| \le 2^p} |\hat{f}(n)| = \sum_{2^{p-1} < |n| \le 2^p} |\hat{f}(n)| \cdot 1$$

$$\le \sqrt{\sum_{2^{p-1} < n \le 2^p} 1} \cdot \sqrt{\sum_{2^{p-1} < n \le 2^p} |\hat{f}(n)|^2}$$

$$\le 2^{\frac{p-1}{2}} \frac{\sqrt{2} K \pi^\alpha}{2^{(p+1)\alpha}}$$

$$= C_\alpha \cdot \frac{2^{p/2}}{2^{p\alpha}} = C_\alpha \frac{1}{2^{p(\alpha - \frac{1}{2})}}$$

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{p=1}^{\infty} \sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)| + |\hat{f}(0)|$$

$$\leq \underbrace{C_\alpha \sum_{p=1}^{\infty} \frac{1}{2^{p(\alpha - \frac{1}{2})}}}_{< \infty} + |\hat{f}(0)|$$

$\because \ \alpha > 1/2$

1. (Wirtinger's inequality) Let f be a T-periodic, continuous and piecewise C^1 function.

   (a) Suppose
$$\int_0^T f(t)dt = 0.$$
   Show that
$$\int_0^T |f(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt,$$
   with equality holds if and only if f takes the form $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$.

General Setting :

Let
$$f : [a,b] \longrightarrow \mathbb{C}$$

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x/L} dx \qquad (L := b-a)$$

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x/L}$$

The functions $f(x)$ and $S_N f(x)$ have the same Fourier coefficients for $|n| \leq N$

Note also $\int_a^b e^{2\pi i n x/L} e^{-2\pi i m x/L} dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

If we put
$$\langle f, g \rangle = \frac{1}{L} \int_a^b f(x)\overline{g(x)} dx, \text{ then}$$

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

Note that

$$\int_0^T f(t)\,dt = 0 \iff \hat{f}(0) = 0$$

We compare $\int_0^T |f(t)|^2\,dt$ and $\int_0^T |f'(t)|^2\,dt$

by the Fourier coefficients of f and f'.

$$\widehat{f'}(n) = \frac{1}{T}\int_0^T f'(t)\,e^{-2\pi i n t/T}\,dt$$

$$= \frac{1}{T}\int_0^T \left(f(t)\,e^{-2\pi i n t/T}\right)' + \frac{2\pi i n}{T} f(t)\,e^{-2\pi i n t/T}\,dt$$

$$= \frac{1}{T}(0) + \frac{2\pi i n}{T}\hat{f}(n)$$

(1st Fundamental Thm of Calculus,
$f(t)\,e^{-2\pi i n t/T}$ is T-periodic)

Note that $\widehat{f'}(0) = 0$

$$\frac{1}{T}\int_0^T |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

$$= \sum_{n\neq 0} \frac{T^2}{4\pi^2 n^2}|\widehat{f'}(n)|^2$$

$$\boxed{\leq} \ \frac{T^2}{4\pi^2} \sum_{n \neq 0} |\hat{f'}(n)|^2$$

$$= \frac{T^2}{4\pi^2} \sum_{n=-\infty}^{\infty} |\hat{f'}(n)|^2$$

$$= \frac{T^2}{4\pi^2} \left(\frac{1}{T} \int_0^T |f'(t)|^2 dt \right)$$

$$\therefore \quad \int_0^T |f(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt$$

"$=$" iff $\hat{f}(n) = 0$ for $|n| > 1$

In this case, $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$

and $S_N f \rightrightarrows f$ (since f is cont)

$$\therefore \quad f(x) = \cancel{\hat{f}(0)}^{\,0} + \hat{f}(1) e^{2\pi i x/T} + \hat{f}(-1) e^{-2\pi i x/T}$$

i.e.

$$f(x) = A \sin\left(2\pi x/T\right) + B \cos\left(2\pi x/T\right)$$

It is easy to see that if f is given like this, then $\hat{f}(n) = 0$ for $|n| > 1$ #

(b) Hence concludes that, in general,

$$\int_0^T |f(t) - \bar{f}|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt,$$

where $\bar{f} = \frac{1}{T} \int_0^T f(t)dt$. When does equality hold in this case?

Let $g(t) = f(t) - \bar{f}$, and check that function g satisfies the assumption in 1(a).

2. Let f, g be T-periodic, continuous and piecewise C^1 functions. Suppose

$$\int_0^T f(t)dt = 0.$$

Show that
$$\left|\int_0^T \overline{f(t)}g(t)dt\right|^2 \leq \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2 dt \int_0^T |g'(t)|^2 dt.$$

Let $G(t) = g(t) - \frac{1}{T}\int_0^T g(x)dx$

$= g(t) - \bar{g}$

$$\int_0^T |G(t)|^2 dt \leq \frac{T^2}{4\pi^2}\int_0^T |g'(t)|^2 dt \quad \text{— by (b)}$$

$$\left|\int_0^T \overline{f(t)}G(t)\,dt\right|^2 \leq \int_0^T |G(t)|^2 dt \cdot \int_0^T |f(t)|^2 dt$$

$$\leq \frac{T^2}{4\pi^2}\int_0^T |g'(t)|^2 dt \int_0^T |f(t)|^2 dt$$

$$\int_0^T \overline{f(t)}G(t)\,dt = \int_0^T \overline{f(t)}g(t)dt + \int_0^T \overline{f(t)}\,\boxed{\bar{g}}\,dt$$

$$= \int_0^T \overline{f(t)}g(t)dt + 0$$

A constant

3. Let f be a C^1 function on $[a,b]$ such that $f(a) = f(b) = 0$. Show that

$$\int_a^b |f(t)|^2 dt \le \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2 dt.$$

When does equality hold?

Proof of I(a) yields

$$\int_0^T |f(t)|^2 dt \le \frac{T^2}{4\pi^2} \int_0^T |f'(t)|^2 dt + T|\hat{f}(0)|^2$$

Doesn't work!

We will make up a fcn g s.t.

(i) g inherits f

(ii) $\hat{g}(0) = 0$

Let $g : [0,T] \to \mathbb{R}$ s.t. $(T = b-a)$

$$g(x) = f(a+x)$$

and extend g as a $2T$-periodic odd fcn on $\mathbb{R}$

i.e.
$$g(x) = \begin{cases} f(a+x) & \text{for } x \in [0,T] \\ -f(a-x) & \text{for } x \in [-T,0] \end{cases}$$

Then, $g'(x) = f'(a+x)$

$$\Rightarrow \int_0^{2T} |g(t)|^2 dt \le \frac{(2T)^2}{4\pi^2} \int_0^{2T} |g'(t)|^2 dt$$

$$= \frac{T^2}{\pi^2} \int_0^{2T} |f'(t)|^2 dt$$

$$= \frac{2T^2}{\pi^2} \int_0^{T} |f'(t)|^2 dt$$

i.e. $\int_0^T |f(t)|^2 dt \le \frac{T^2}{\pi^2} \int_0^T |f'(t)|^2 dt$

"=" iff $g(x) = A \sin\left(\frac{2\pi x}{2T}\right)$ (g is odd fcn)

$= A \sin\left(\frac{\pi x}{T}\right)$

i.e. $f(x) = ??$